1.6. Properties of the complex coefficient of pair correlation

Having determined how to calculate correctly the complex coefficient of pair correlation between two random complex variables, one should give an interpretation to the values that it can take.

A linear relationship between two complex variables means that both the real and imaginary parts of one complex variable act as two-factor linear dependencies on the real and imaginary parts of another complex variable. Therefore, if one variable changes non-linearly, then the other variable will change non-linearly, and such a dependence is often difficult to be determined visually. If the dependence under study is not functional, but regression, then the scatter of points on the complex planes causes is even less associated with any dependence. Therefore, a visual analysis of the relationship between complex variables is difficult and the linear relationship of two complex variables can be judged solely by the calculated characteristics. In the domain of real variables, this function is performed by the pair correlation coefficient, and in the domain of complex random variables, by the complex pair correlation coefficient (1.5.11).

The complex coefficient of pair correlation, as it follows from the materials of the previous paragraph, is the geometric mean of two complex regression coefficients

$$r_{XY} = \pm\sqrt{ab} \quad (1.6.1)$$

Therefore, for a strictly functional linear complex-valued dependence

$Y = aX$, $X = bY$ the obvious equality will hold:

$$a = \frac{1}{b}, \quad \text{whence:} \quad ab = \frac{1}{b}b = 1 \quad (1.6.2)$$

That is, the complex coefficient of pair correlation for a linear functional dependence is equal to:

$$r_{XY} = \pm(1 + i0) \quad (1.6.3)$$

So, for a linear functional relationship between two complex variables, the modulus of the real part of the complex coefficient of pair correlation

will be equal to one, and its imaginary component will be zero.

This means that the square of the complex coefficient of pair correlation (complex coefficient of determination) for a linear relationship will be always equal to a real unit. But in which cases of linear functional dependence between two complex variables will the correlation coefficient take the values "plus one", and in which cases – "minus one"?

To answer this question, we present the complex proportionality coefficients in exponential and trigonometric forms:

$$a = R_a e^{i\alpha} = R_a[\cos\alpha + i\cos\alpha], \quad (1.6.4)$$

$$b = R_b e^{i\beta} = R_b[\cos\beta + i\cos\beta]\ , \quad (1.6.5)$$

Then their product will be equal to:

$$ab = R_a R_b e^{i(\alpha+\beta)} = R_a R_b[\cos(\alpha+\beta) + i\cos(\alpha+\beta)]\ . \quad (1.6.6)$$

Since for a linear functional dependence (1.6.3) holds, that is, the imaginary part of the complex coefficient of pair correlation is zero, hence it clearly follows that in this case the condition

$$\alpha + \beta = 2\pi k, \quad k = 0,1,2,...\ . \quad (1.6.7)$$

must be met.

We will consider for simplicity the case when *k*=0. Then the complex pair correlation coefficient is defined as the square root of such a product:

$$r_{cXY} = \pm\sqrt{ab} = \pm\sqrt{R_a R_b \cos(\alpha+\beta)}\ . \quad (1.6.8)$$

Since the modulus of each proportionality coefficient is positive by definition, the equality of the complex coefficient of pair correlation

"plus one" or "minus one" is completely determined by the cosine of the angle α. We are interested in the case when a radicand can be like this:

$$\sqrt{(-1)(-1)} \qquad (1.6.9)$$

Then it is possible to determine the characteristics of dependence in the case when the complex coefficient of proportionality becomes equal to minus one. From the equality (1.6.7), the cosine of the radicand (1.6.8) can be written as:

$$\cos(\alpha + (-\alpha)) = \cos\alpha\cos(-\alpha) - \sin\alpha\sin(-\alpha)\ ,$$

Now it is easy to determine that the case of interest to us (1.6.9) is determined by the polar angle of the proportionality complex coefficient $a_0 + ia_1$, lying in the range:

$$\frac{3}{4}\pi \le \alpha \le \frac{5}{4}\pi \qquad (1.6.10)$$

For this case, the real component of the complex proportionality coefficient is always not positive:

$$a_0 \le 0, \qquad (1.6.11)$$

and its imaginary part is always not less than the real one:

$$a_0 \le a_1 \qquad (1.6.12)$$

These conditions are satisfied, for example, by such complex proportionality coefficients:

$-1+i; \ -1+0i; \ -10-i9,999...$

What is a linear complex-valued relationship with such values?

To answer this question, we present a linear functional complex-valued dependence as a system of two equalities of real and imaginary parts:

$$y_r = a_0 x_r - a_1 x_i ,$$

(1.6.13)

$$y_i = a_1 x_r + a_0 x_i , \quad (1.6.14)$$

According to the conditions (1.6.11) and (1.6.12), the coefficient a_0 is always not positive, and the imaginary part a_1 can take both positive and negative values. Considering the situation when the complex argument increases in the first quadrant of the complex plane, that is, when x_r and x_i are continuously growing and are positive numbers, we obtain that in this case the real part of the complex result Y_r decreases, and the imaginary part Y_i because of (1.6.12) can both increase and decrease

That is, the values equal to "minus one" are accepted by the complex coefficient of pair correlation only when there is an inverse relationship between the variables – an increase in the values of the argument X results in a decrease in the values of the real part Y_r of the complex random variable *Y*.

So, the real part of the complex coefficient of pair correlation r_r indicates the degree of the dependence approximation between random complex variables to a linear dependence and the interpretation of its values is similar to the values interpretation of the pair correlation coefficient in the real numbers' domain.

The imaginary component of the complex coefficient of pair correlation r_i, as it clearly follows from (1.6.3), in the case when there is a linear functional dependence between the complex variables, will be zero.

So, if the real part of the complex coefficient of pair correlation modulo is close to one, and its imaginary part modulo is close to zero, then the researcher can claim that the dependence, the presence of which he assumes between two complex random variables, is close to linear.